Almost Complex Structure on S^{2n}

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Abstract

We show that there is no complex structure in a neighborhood of the space of orthogonal almost complex structures on the sphere S^{2n}, $n > 1$. The method is to study the first Chern class of vetcor bundle $T^{(1,0)}S^{2n}$.

In [4], we showed that the orthogonal twistor space $\widetilde{\mathcal{J}}(S^{2n})$ of the sphere S^{2n} is a Kaehler manifold and an orthogonal almost complex structure J_f on S^{2n} is integrable if and only if the corresponding section f: $S^{2n} \to \widetilde{\mathcal{J}}(S^{2n})$ is holomorphic. These shows there is no orthogonal complex structure on the sphere S^{2n} for $n > 1$.

We can show that the first Chern class of the vector bundle $T^{H(1,0)}\widetilde{\mathcal{J}}(S^{2n})$ can be represented by the Kaehler form of $\widetilde{\mathcal{J}}(S^{2n})$ with a constant as coeifficient. Thus $c_1(T^{(1,0)}S^{2n}) = f^*c_1(T^{H(1,0)}\widetilde{\mathcal{J}}(S^{2n})) \in H^2(S^{2n})$ is non-zero if the map f: $S^{2n} \to \widetilde{\mathcal{J}}(S^{2n})$ is holomorphic. In following we show that there is no complex structure in a neighborhood of the twistor space $\widetilde{\mathcal{J}}(S^{2n})$, the method is to study $c_1(T^{(1,0)}S^{2n})$.

Let $\langle\, ,\, \rangle$ be the canonical Riemanian metric on the sphere S^{2n} and J_f be an almost complex structure.

$$\mathrm{d}s_f^2(X, Y) = \langle X, Y\rangle_f = \frac{1}{2}\langle X, Y\rangle + \frac{1}{2}\langle J_f X, J_f Y\rangle, \quad X, Y \in TS^{2n} \otimes \mathbf{C}$$

defines a Hermitian metric on $TS^{2n} \otimes \mathbf{C}$. Let $\tilde{e}_1, \cdots, \tilde{e}_{2n}$ be local $\langle\, ,\, \rangle$-orthonormal frame fields on S^{2n}, $\tilde{\omega}^1, \cdots, \tilde{\omega}^{2n}$ be their dual. The almost complex structure J_f can be represented by $J_f = \sum\ \tilde{e}_i J_{ij}\tilde{\omega}^j$, $J_f(X) = \sum\ \tilde{e}_i J_{ij}\tilde{\omega}^j(X)$, the Hermitian metric can be represented by

$$\mathrm{d}s_f^2 = \frac{1}{2}\sum\ (\delta_{ij} + \sum\ J_{ki}J_{kj})\tilde{\omega}^i \otimes \tilde{\omega}^j.$$

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Remark Let (P_{ij}) be the positive definite symmetric matrix such that $(P_{ij})^2 = \frac{1}{2}(\delta_{ij} + \sum\ J_{ki}J_{kj})$. The tensor field $P = \sum\ \tilde{e}_i P_{ij} \tilde{\omega}^j$ is well defined. For any $\langle\ ,\ \rangle_f$-orthogonal almost complex structure J_1 and $X, Y \in TS^{2n}$, we have

$$\langle PX, PY \rangle = \langle X, Y \rangle_f,$$

$$\langle PJ_1P^{-1}X, PJ_1P^{-1}Y \rangle = \langle J_1P^{-1}X, J_1P^{-1}Y \rangle_f = \langle X, Y \rangle.$$

These shows the almost complex structure PJ_1P^{-1} preserves the metric $\langle\ ,\ \rangle$.

By the proof of Theorem 2.4 in [4], there are local $\langle\ ,\ \rangle$-orthonormal frame fields $\bar{e}_1, \bar{e}_2, \cdots, \bar{e}_{2n}$ and their dual $\bar{\omega}^1, \cdots, \bar{\omega}^{2n}$ such that

$$J_f = \sum\ \sqrt{1+\lambda_i^2}(\bar{e}_{2i}\bar{\omega}^{2i-1} - \bar{e}_{2i-1}\bar{\omega}^{2i}) + \sum\ \lambda_i(\bar{e}_{2i-1}\bar{\omega}^{2i-1} - \bar{e}_{2i}\bar{\omega}^{2i}).$$

Set $\tilde{e}_{2i-1} = \frac{\sqrt{2}}{2}(\bar{e}_{2i-1} + \bar{e}_{2i}),\ \tilde{e}_{2i} = \frac{\sqrt{2}}{2}(-\bar{e}_{2i-1} + \bar{e}_{2i})$, we have

$$J_f = \sum\ (\lambda_i + \sqrt{1+\lambda_i^2})\tilde{e}_{2i}\tilde{\omega}^{2i-1} + \sum\ (\lambda_i - \sqrt{1+\lambda_i^2})\tilde{e}_{2i-1}\tilde{\omega}^{2i}.$$

Then we have $P = \sum_{k=1}^{2n}\ \mu_k \tilde{e}_k \tilde{\omega}^k$, where

$$\mu_{2i-1} = \left(1 + \lambda_i^2 + \lambda_i\sqrt{1+\lambda_i^2}\right)^{\frac{1}{2}},\ \mu_{2i} = \left(1 + \lambda_i^2 - \lambda_i\sqrt{1+\lambda_i^2}\right)^{\frac{1}{2}}.$$

Set $e_k = \frac{1}{\mu_k}\tilde{e}_k$, we have $J_f(e_{2i-1}) = e_{2i},\ (PJ_fP^{-1})(\tilde{e}_{2i-1}) = \tilde{e}_{2i}$. $\square$

Let ∇^f be the Riemannian connection of the metric $\langle\ ,\ \rangle_f$ and $e_1, \cdots, e_{2n}$ be J_f-frame fields, $\langle e_i, e_j \rangle_f = \delta_{ij},\ J_f e_{2i-1} = e_{2i},\ \omega^1, \cdots, \omega^{2n}$ be their dual, $J_f\omega^{2i-1} = -\omega^{2i}$. Let

$$\nabla^f (e_1, \cdots, e_{2n-1}, e_2, \cdots, e_{2n})^t = \omega(e_1, \cdots, e_{2n-1}, e_2, \cdots, e_{2n})^t,$$

where $\omega = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ be the connection matrix, $\Omega = d\omega - \omega \wedge \omega$ the curvature matrix. $B + C = A - D = 0$ if J_f is integrable and ∇^f a Kaehler connection.

Lemma 1 J_f is a complex structure if and only if $J_f(B + C) = A - D$.

Proof Let $Z_i = e_{2i-1} - \sqrt{-1}e_{2i},\ Z_{\bar{i}} = e_{2i-1} + \sqrt{-1}e_{2i}$ be the $(1,0)$ and $(0,1)$ frame fields on S^{2n}, $i = 1, \cdots, n$. Then we have

$$\begin{aligned}
\nabla^f \begin{pmatrix} Z_i \\ Z_{\bar{i}} \end{pmatrix} &= \nabla^f \begin{pmatrix} I & -\sqrt{-1}I \\ I & \sqrt{-1}I \end{pmatrix} \begin{pmatrix} e_{2i-1} \\ e_{2i} \end{pmatrix} \\
&= \frac{1}{2} \begin{pmatrix} I & -\sqrt{-1}I \\ I & \sqrt{-1}I \end{pmatrix} \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} I & I \\ \sqrt{-1}I & -\sqrt{-1}I \end{pmatrix} \begin{pmatrix} Z_i \\ Z_{\bar{i}} \end{pmatrix} \\
&= \frac{1}{2} \begin{pmatrix} A + D + \sqrt{-1}(B - C) & A - D - \sqrt{-1}(B + C) \\ A - D + \sqrt{-1}(B + C) & A + D - \sqrt{-1}(B - C) \end{pmatrix} \begin{pmatrix} Z_i \\ Z_{\bar{i}} \end{pmatrix}.
\end{aligned}$$

The vector bundle $T^{(1,0)}S^{2n}$ is generated by $Z_1, \cdots, Z_n$. As shown in [1], [4], the almost complex structure J_f is integrable if and only if $\nabla^f_X Y \in \Gamma(T^{(1,0)}S^{2n})$ for any $X, Y \in \Gamma(T^{(1,0)}S^{2n})$. Then J_f is integrable if and only if $A - D - \sqrt{-1}(B+C)$ are formed by $(0,1)$-forms, that is

$$J_f(B+C) = A - D. \qquad \square$$

Lemma 2 The first Chern class of $T^{(1,0)}S^{2n}$ can be represented by

$$c_1(T^{(1,0)}S^{2n}) = -\frac{1}{4\pi}\mathrm{tr}(\Omega J_0 + \omega \wedge \omega J_0),$$

where $J_0 = \begin{pmatrix} & -I \\ I & \end{pmatrix}$.

Proof With notations used above, the induced connection ∇^f on $T^{(1,0)}S^{2n}$ is

$$\nabla^f Z_i = \sum \ \psi_i^k Z_k,$$

where $\psi_i^k = \frac{1}{2}\sum\ (\omega_{2i-1}^{2k-1} + \omega_{2i}^{2k} + \sqrt{-1}(\omega_{2i-1}^{2k} - \omega_{2i}^{2k-1}))$ and the curvature forms $\Psi_i^k = \mathrm{d}\psi_i^k - \sum\ \psi_i^j \wedge \psi_j^k$, $\psi_i^k + \overline{\psi}_k^i = 0$, $\Psi_i^k + \overline{\Psi}_k^i = 0$. The first Chern class of the vector bundle $T^{(1,0)}S^{2n}$ can be represented by

$$c_1(T^{(1,0)}S^{2n}) = \frac{\sqrt{-1}}{2\pi}\sum\ \Psi_i^i.$$

By $\sum\ \psi_i^j \wedge \psi_j^i = -\sum\ \psi_j^i \wedge \psi_i^j = 0$ and $\sum\ \mathrm{d}\omega_{2i-1}^{2i} = \frac{1}{2}\mathrm{tr}(\mathrm{d}\omega J_0) = \frac{1}{2}\mathrm{tr}(\Omega J_0 + \omega \wedge \omega J_0)$, we have

$$c_1(T^{(1,0)}S^{2n}) = -\frac{1}{4\pi}\mathrm{tr}(\Omega J_0 + \omega \wedge \omega J_0). \qquad \square$$

As [4], let $\mathcal{J}(S^{2n})$ be the twistor space on S^{2n}, its sections are the almost complex structures on S^{2n}.

Theorem 3 When $n > 1$, there is no complex structure in a neighborhood of the space $\widetilde{\mathcal{J}}(S^{2n})$.

Proof By

$$\omega + J_0 \omega J_0 = \begin{pmatrix} A - D & B + C \\ B + C & -A + D \end{pmatrix}, \ J_0\omega - \omega J_0 = \begin{pmatrix} -B - C & A - D \\ A - D & B + C \end{pmatrix},$$

we see that the equation $J_f(B+C) = A - D$ is equivalent to $J_f(\omega + J_0\omega J_0) = J_0(\omega + J_0\omega J_0)$. Then, if J_f is integrable, we have

$$\begin{aligned} & \mathrm{tr}(\omega \wedge \omega J_0) \\ = & \frac{1}{4}\mathrm{tr}[(\omega + J_0\omega J_0) \wedge (\omega J_0 - J_0\omega)] \\ = & -\mathrm{tr}[(B+C) \wedge J_f(B+C)] \\ = & \mathrm{tr}[(B+C) \wedge J_f(B+C)^t]. \end{aligned}$$

The sectional curvature of the metric $\langle\, ,\, \rangle$ on S^{2n} is constant, $\Omega_k^l = -\omega^k \wedge \omega^l$. Then if J_f is a $\langle\, ,\, \rangle$-orthogonal complex structure, we have

$$\begin{aligned}
&\mathrm{tr}(\Omega J_0) + \mathrm{tr}(\omega \wedge \omega J_0) \\
&= \sum\ 2\omega^{2i-1} \wedge J_f \omega^{2i-1} + \sum\ (\omega_{2i-1}^{2j} + \omega_{2i}^{2j-1}) \wedge J_f(\omega_{2i-1}^{2j} + \omega_{2i}^{2j-1}).
\end{aligned}$$

For any $X \in TS^{2n}$, we have

$$\begin{aligned}
&[\mathrm{tr}(\Omega J_0) + \mathrm{tr}(\omega \wedge \omega J_0)](X, J_f X) \\
&= -\sum\ 2([\omega^{2i-1}(X)]^2 + [\omega^{2i}(X)]^2) \\
&\quad -\sum\ [(\omega_{2i-1}^{2j} + \omega_{2i}^{2j-1})(X)]^2 - \sum\ [(\omega_{2i-1}^{2j-1} - \omega_{2i}^{2j})(X)]^2.
\end{aligned}$$

Then 2-form $\mathrm{tr}(\Omega J_0 + \omega \wedge \omega J_0)$ are non-degenerate everywhere and S^{2n} becomes a symplectic manifold, this contradict to the fact of $H^2(S^{2n}) = 0$ for $n > 1$. As the Riemannian curvature is continuous with the Riemannian metric, these shows there is a neighborhood of $\widetilde{\mathcal{J}}(S^{2n})$ in $\mathcal{J}(S^{2n})$ such that there is no complex structure in this neighborhood. $\square$

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